Afdeling Wiskunde en Informatica R.U.G.

Bladnr.: 1 van 4
Tentamen: Metrische ruimten
Datum: 28-06-2004
Naam docent: Winkler.

Exercise 2:

$\frac{2}{2{,}5}$

Let A be an infinite set with some element $p \in A$. Let τ consist of all subsets U of A such that $p \in U$, together with the empty set $\emptyset$.

2) a). Show $T = (A, \tau)$ is a topological space.
Three things must be true:

1). • $\emptyset, A \in \tau$.
2). • $U_1 \cap U_2 \in \tau$ for any $U_1, U_2 \in \tau$.
3). • $\bigcup_{i \in I} U_i \in \tau$ for any $U_i \in \tau$ and any index set I

1). $p \in A \Rightarrow A \in \tau$ and $\emptyset \in \tau$ as well, so this condition is satisfied
2). take $U_1, U_2 \in \tau$. $p \in U_1$ and $p \in U_2 \therefore p \in U_1 \cap U_2$. ~~the~~ ~~therefore~~ Therefore $U_1 \cap U_2 \in \tau$.
3). $p \in U_i$ for all i. Therefore, in every union containing any U, p is in the union, therefore $\bigcup_{i \in I} U_i \in \tau$.

✓

2).b). Is T compact?
Compactness means that given any cover of T there exists a finite subcover.

T is not compact. ✓

~~take the cover 𝒰 with elements (U_n) with $U_n \subset U_{n+1}$ there is no finite subcover because A is infinite and for a U_i to cover all of it, it (must) must be infinite so the distance from p to the~~

Take for example the set $B_1(0)$ in $\mathbb{R}^2$, with topology $\{B_\varepsilon(0) \mid 0 \le \varepsilon \le 1\}$. A cover for $B_1(0)$ would be $\{B_\varepsilon(0) \mid 0 < \varepsilon < 1\}$, but there is no possible finite subcover.

✓ Why not? (details)

2). c). Is T connected?

T is connected if the only sets which are both open and closed are A and $\emptyset$.

take $U \in \mathcal{T}$. It is open because it is in $\mathcal{T}$. Is it also closed? Then $A \setminus U$ would have to be open. But $p \in U$ and $p \notin A \setminus U$, therefore $A \setminus U \notin \mathcal{T}$, so $A \setminus U$ is closed. Therefore T is connected because no set except T and $\emptyset$ are open and closed.

2). d). Is T Hausdorff?

T is Hausdorff if given $x, y \in A$, $x \neq y$ $\exists U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

T is not Hausdorff: If we take x, y and two sets $U, V \in \mathcal{T}$ with $x \in U$ and $y \in V$, the element p is in both U and V (by definition) therefore there is at least one element in $U \cap V$, that is p. Therefore T is not Hausdorff.

2). e). Can $\mathcal{T}$ be generated by a metric defined on A?

~~(yes, by $d(x,y) = |x-p| + |y-p|$)~~

yes. no, (not Hausd.) $(-\frac{1}{2})$

Afdeling Wiskunde en Informatica R.U.G.

Naam:
Adres:
Postcode en
Woonplaats:

Studentnummer:
Studierichting:

Jaar van eerste inschrijving:

Bladnr.: 2 v.d. 4
Tentamen:
Datum:
Naam docent:

Exercise 3.

2/2

Let the topological space T be Hausdorff. Show that the finite subsets of T are closed.

~~(Take A_1, A_2 finite)~~

Take $T = (A, \tau)$, with τ the topology and A the set.

Hausdorff means that if we take $x, y \in A$, $\exists\, U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

It is enough to show that $\{x\}$ for any $x \in A$ is closed because the finite union of closed sets is closed.

~~$A \setminus \{x\}$ is Hausdorff because it is the subspace of a Hausdorff space~~

$A \setminus \{x\}$ is also Hausdorff. We therefore know that if we take $n, m \in A \setminus \{x\}$, then exist open U, V with $n \in U$, $m \in V$ and $U \cap V = \emptyset$. ~~(Therefore we have an)~~ This is so for any point n and m, therefore we have proved that around any point of $A \setminus \{x\}$ we can construct an open subset of $A \setminus \{x\}$, which implies that $A \setminus \{x\}$ is open. We therefore know that $\{x\}$ is closed, therefore all finite subsets of A are closed.

Afdeling Wiskunde en Informatica R.U.G.

Naam:	Studentnummer:	Bladnr.: 3 v/d 4
Adres:	Studierichting:	Tentamen:
Postcode en		Datum:
Woonplaats:	Jaar van eerste inschrijving:	Naam docent:

Exercise 4: Determine closure and boundary of each of the following subsets of $\mathbb{R}$ with the Euclidean metric. Which of these sets are dense or nowhere dense in $\mathbb{R}$?

2,5
2,5

closure: ~~union of H~~ $Cl(H)$ = union of H with all its limit points

limit points: any point x with $(U \setminus \{x\}) \cap \mathbb{R} \neq \emptyset$ for all open sets U containing x.

boundary: boundary of $H = Cl(H) \cap Cl(\mathbb{R} \setminus H)$.

nowhere dense: $Int(Cl(H)) = \emptyset$.

dense: if $\mathbb{R} \setminus H$ is nowhere dense.
if $Cl(H) = \mathbb{R}$.

4).a). let subset $H = \mathbb{R}$.

closure $= \mathbb{R}$, boundary $= Cl(\mathbb{R}) \cap Cl(\mathbb{R} \setminus \mathbb{R}) = \emptyset$,

$\mathbb{R}$ is dense in $\mathbb{R}$ because $\mathbb{R} \setminus \mathbb{R} = \emptyset$, and $Int(Cl(\emptyset)) = \emptyset$.

4).b). $H = \mathbb{Q} \cap (-\infty, 0)$.

closure $= (-\infty, 0]$, because for any $a \in (-\infty, 0)$ we can always get closer and closer to it with elements from $\mathbb{Q}$.

boundary $= Cl(\mathbb{Q} \cap (-\infty, 0)) \cap Cl(\mathbb{R} \setminus (\mathbb{Q} \cap (-\infty, 0)))$.

$= (-\infty, 0] \cap \mathbb{R}$

~~$Cl(H) \neq \mathbb{R}$ therefore~~

$Int(Cl(H)) = Int((-\infty, 0]) \neq \emptyset$ therefore $\mathbb{Q} \cap (-\infty, 0)$ is dense in $(-\infty, 0]$! (not in R)

4).c). $H = \{\frac{7}{3n} \mid n \in \mathbb{N}\}$

The only limit point is 0, therefore $Cl(H) = \{\frac{7}{3n} \mid n \in \mathbb{N}\} \cup \{0\}$,

boundary $= Cl(H) \cap Cl(\mathbb{R} \setminus H) = (\{\frac{7}{3n}\} \cup \{0\}) \cap \mathbb{R} = \{\frac{7}{3n}\} \cup \{0\}$,

$Int(Cl(H)) = \emptyset$ therefore $\{\frac{7}{3n} \mid n \in \mathbb{N}\}$ is nowhere dense in $\mathbb{R}$.

4).d). $H = \mathbb{R} \setminus \mathbb{Q}$.

Closure: $\mathbb{R} \cap B_\varepsilon(p/q) \neq \emptyset \quad \forall \varepsilon \therefore$ all $\mathbb{Q}$ are limit points

$Cl(H) = \mathbb{R}$,

boundary $= Cl(H) \cap Cl(\mathbb{R} \setminus H) = Cl(\mathbb{R} \setminus \mathbb{Q}) \cap Cl(\mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Q}))$

$= \mathbb{R} \cap Cl(\mathbb{Q}) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$,

$Cl(H) = \mathbb{R}$ therefore $\mathbb{R} \setminus \mathbb{Q}$ is everywhere dense.

4).e). $H = \mathbb{R} \setminus \mathbb{Z}$

Closure $Cl(H) = \mathbb{R}$,

boundary $= Cl(H) \cap Cl(\mathbb{R} \setminus H) = \mathbb{R} \cap Cl(\mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Z})) = \mathbb{R} \cap Cl(\mathbb{Z})$

$= \mathbb{Z}$,

$Cl(H) = \mathbb{R}$ therefore $\mathbb{R} \setminus \mathbb{Z}$ is everywhere dense.

4). f). $H = \mathbb{N}$.

$Cl(\mathbb{N}) = \mathbb{N}$,

boundary $= Cl(\mathbb{N}) \cap Cl(\mathbb{R} \setminus \mathbb{N}) = \mathbb{N} \cap \mathbb{R} = \mathbb{N}$,

$Int(Cl(\mathbb{N})) = Int(\mathbb{N}) = \emptyset \therefore \mathbb{N}$ is nowhere dense in $\mathbb{R}$.

Afdeling Wiskunde en Informatica R.U.G.

9,5/10 P.

Naam:	Studentnummer:	Bladnr.: 4 vd. 4
Adres:	Studierichting:	Tentamen:
Postcode en		Datum:
Woonplaats:	Jaar van eerste inschrijving:	Naam docent:

Exercise 1.:

2/2

Let B be a totally bounded subset of a metric space M. Show that B is bounded. Given an example of a bounded metric space which is not totally bounded.

totally bounded: for every $\varepsilon > 0$ $\exists$ finite ε-net for M.
ε-net: given $\varepsilon > 0$, an ε-net is a subset S of B such that

$B \subseteq \bigcup_{s \in S} B_\varepsilon(s)$. with B_ε open balls.

If B is finite, we can cover each point with a separate ε-ball. Then take $k = \max_{i,j} \{d(x_i, x_j)\}$ for all $x_i, x_j \in B$, ~~and take any x~~.
then all points of B are inside $B_{k+1}(x)$ for any x, therefore B is bounded. (*)

If B is infinite, there exists at least one ε-ball in the ε-net which contains infinitely many points. This part of B is then bounded, with $k = \varepsilon$.
Once we have only ε-balls left which contain finitely many points of B, the reasoning goes as in (*)
Therefore B is made up of the union of finitely many bounded sets, which means that B itself is bounded.

example of bounded metric space which is not totally bounded:

interval in $\mathbb{R}$.

$M = (A, d)$ with $A = (0,1)$ and $d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$

~~[illegible]~~ This metric space is bounded, but it is not totally bounded, if we take $0 < \varepsilon < 1$, infinitely many ε-balls are needed to cover M.